F1 May 16 Model Solutions IAL

Kprime 2

Leave blank

1. Use the standard results for $\sum_{r=1}^{n} r$ and for $\sum_{r=1}^{n} r^3$ to show that, for all positive integers n,

$$\sum_{r=1}^{n} r(r^2 - 3) = \frac{n}{4}(n+a)(n+b)(n+c)$$

where a, b and c are integers to be found.

(4)

$$\text{LHS} = \sum_{r=1}^{n} r(r^2-3) = \sum_{r=1}^{n} r^3 - 3r$$

$$= \sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r$$

$$= \frac{n^2}{4}(n+1)^2 - \frac{3n}{2}(n+1)$$

$$= \frac{n}{4}(n+1)\left[n(n+1) - 6\right]$$

$$= \frac{n}{4}(n+1)(n^2+n-6)$$

$$= \frac{n}{4}(n+1)(n+3)(n-2)$$

$a = 1$

$b = 3$

$c = -2$

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

P 4 6 6 8 3 A 0 2 3 2

Leave blank

2. A parabola P has cartesian equation $y^2 = 28x$. The point S is the focus of the parabola P.

(a) Write down the coordinates of the point S. **(1)**

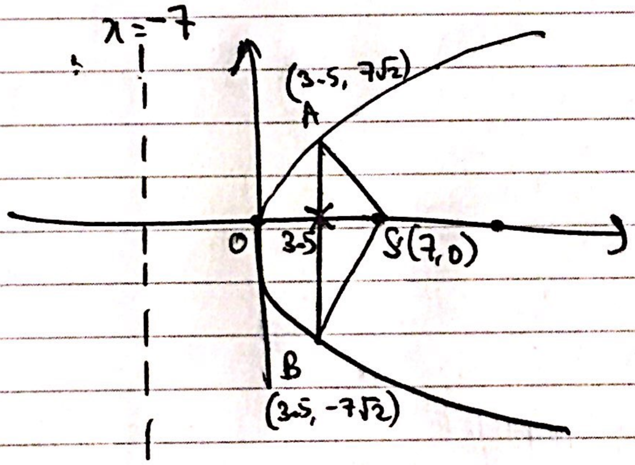
Points A and B lie on the parabola P. The line AB is parallel to the directrix of P and cuts the x-axis at the midpoint of OS, where O is the origin.

(b) Find the exact area of triangle ABS. **(4)**

(a) $y^2 = 4 \times 7x$

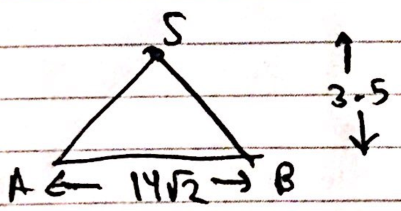
$S: (7, 0)$

(b)



directrix: $x = -7$

$x = 3.5 \Rightarrow y^2 = 98 \Rightarrow y = \pm 7\sqrt{2}$



$\Rightarrow$ Area $= \frac{1}{2} \times 14\sqrt{2} \times 3.5$

$= \frac{49\sqrt{2}}{2}$

$\therefore$ Area $\Delta ABS = \frac{49\sqrt{2}}{2}$

Leave blank

3. $x^2+3x^{-1}-1$

$$f(x) = x^2 + \frac{3}{x} - 1, \qquad x < 0$$

The only real root, α, of the equation $f(x) = 0$ lies in the interval $[-2, -1]$.

(a) Taking -1.5 as a first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 2 decimal places. **(5)**

(b) Show that your answer to part (a) gives α correct to 2 decimal places. **(2)**

(a) $f'(x) = 2x - 3x^{-2}$

$f(-1.5) = -\frac{3}{4}$

$f'(-1.5) = -\frac{13}{3}$

$\therefore \alpha \approx -1.5 - \dfrac{f(-1.5)}{f'(-1.5)} = -\frac{87}{52}$

$\therefore \alpha \approx -1.67$ (2dp)

(b) $f(-1.665) = -0.02957\ldots$

$f(-1.675) = +0.014580\ldots$

$\therefore$ There's a sign change in the interval $[-1.675, -1.665]$

$\Rightarrow -1.675 < \alpha < -1.665$

$\therefore \alpha \approx -1.67$ (2dp)

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

4. Given that

$$\mathbf{A} = \begin{pmatrix} k & 3 \\ -1 & k+2 \end{pmatrix}, \text{ where } k \text{ is a constant}$$

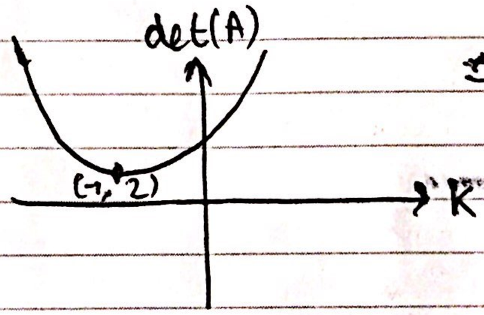
(a) show that $\det(\mathbf{A}) > 0$ for all real values of k, **(3)**

(b) find $\mathbf{A}^{-1}$ in terms of k. **(2)**

(a) $\det(A) = k(k+2) + 3 = k^2 + 2k + 3$

$= (k+1)^2 + 2$

$\Rightarrow$



$\Rightarrow \det(A) > 0$ for all $k \in \mathbb{R}$

Also, $\frac{\partial}{\partial k}(k^2+2k+3) = 2k+2$

$\therefore \frac{\partial}{\partial k}[\det(A)] = 2k+2 = 0$

$\Rightarrow k = -1$

$k = -1 \Rightarrow \det(A) = 2 \Rightarrow$ Min point is $(-1, 2)$ $\} \det(A) \geq 2 > 0$

$\therefore \det(A) > 0$ for all k

(b) $A^{-1} = \frac{1}{k^2+2k+3}\begin{pmatrix} k+2 & -3 \\ 1 & k \end{pmatrix}$

Leave blank

5.

$$2z + z^* = \frac{3+4i}{7+i}$$

Find z, giving your answer in the form $a + bi$, where a and b are real constants. You must show all your working.

(5)

5. RHS $= \dfrac{3+4i}{7+i} = \dfrac{(3+4i)(7-i)}{(7+i)(7-i)}$

$= \dfrac{21+25i-4i^2}{50} = \dfrac{25+25i}{50}$

$= \frac{1}{2} + \frac{1}{2}i$

$\therefore$ LHS $= 2z+z^* = 2(a+bi) + a - bi$

$= 3a + bi$

LHS = RHS $\Rightarrow$ $3a + bi = \frac{1}{2} + \frac{1}{2}i$

$\therefore$ $3a = \frac{1}{2} \Rightarrow a = \frac{1}{6}$

$b = \frac{1}{2}$

$\therefore$ $z = \frac{1}{6} + \frac{1}{2}i$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

6. The rectangular hyperbola H has equation $xy = 25$

(a) Verify that, for $t \neq 0$, the point $P\left(5t, \dfrac{5}{t}\right)$ is a general point on H. **(1)**

The point A on H has parameter $t = \dfrac{1}{2}$

(b) Show that the normal to H at the point A has equation

$$8y - 2x - 75 = 0$$

(5)

This normal at A meets H again at the point B.

(c) Find the coordinates of B. **(4)**

6(a) P has parameters $x = 5t$, $y = \frac{5}{t}$

$\therefore x_p y_p = 5t \times \frac{5}{t} = 25$, $t \neq 0$

$\Rightarrow$ For $P(x_p, y_p)$, $xy = 25$

$\therefore$ P lies on H.

(b) $A\left(\frac{5}{2}, 10\right)$

$y = 25x^{-1} \Rightarrow \frac{\partial y}{\partial x} = -25x^{-2} = -\frac{25}{x^2}$

@ A, $\frac{\partial y}{\partial x} = -\frac{25}{(5/2)^2} = -4$

$\Rightarrow$ Gradient of normal $= \frac{1}{4}$

$\therefore y - y_1 = m(x - x_1) \Rightarrow y - 10 = \frac{1}{4}\left(x - \frac{5}{2}\right)$

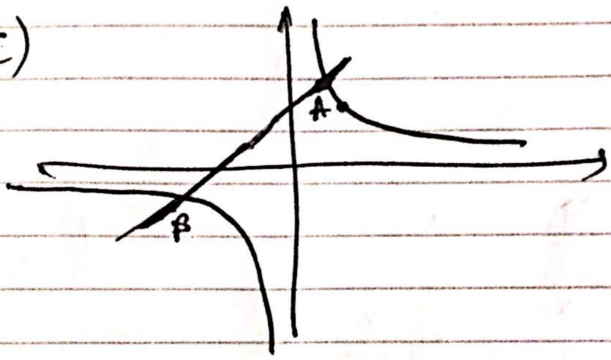
Question 6 continued

$\therefore\ 4y - 40 = x - \frac{5}{2}$

$\times 2 \Rightarrow 8y - 80 = 2x - 5$

$\therefore\ 8y - 2x - 75 = 0$ as required.

(c)



$y = \frac{25}{x} \Rightarrow 8\left(\frac{25}{x}\right) - 2x - 75 = 0$

$\therefore\ \frac{200}{x} - 2x - 75 = 0$

$\times x \Rightarrow 200 - 2x^2 - 75x = 0$

$\therefore\ 2x^2 + 75x - 200 = 0$

$\therefore\ (2x-5)(x+40) = 0$

$x = \frac{5}{2}$ @A $\Rightarrow x_B = -40$

$\therefore\ y_B = \frac{25}{-40} = -\frac{5}{8}$

$\therefore\ B\left(-40, -\frac{5}{8}\right)$

blank

7.

$$\mathbf{P} = \begin{pmatrix} \frac{5}{13} & -\frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation U represented by the matrix **P**. **(3)**

The transformation V, represented by the 2×2 matrix **Q**, is a reflection in the line with equation $y = x$

(b) Write down the matrix **Q**. **(1)**

Given that the transformation $\overset{Q}{V}$ followed by the transformation $\overset{P}{U}$ is the transformation T, which is represented by the matrix **R**,

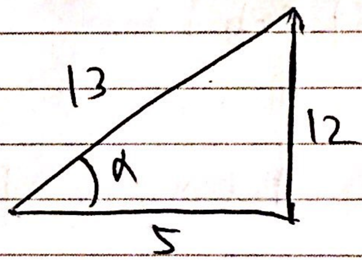
(c) find the matrix **R**. **(2)**

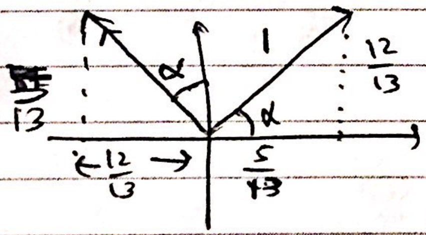
(d) Show that there is a value of k for which the transformation T maps each point on the straight line $y = kx$ onto itself, and state the value of k. **(4)**

(a)

$\tan\alpha = \frac{12}{5}$

$\alpha = 67.4°$ (3sf)





U represents an anticlockwise rotation by $\arctan\left(\frac{12}{5}\right) = 67.4°$ about the origin $(0, 0)$

Question 7 continued

(b)  $\Rightarrow Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(c) $R = PQ$

$\therefore R = \begin{pmatrix} 5/13 & \frac{-12}{13} \\ 12/13 & 5/13 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} -12/13 & 5/13 \\ 5/13 & 12/13 \end{pmatrix}$ //

(d) $R \begin{pmatrix} x \\ kx \end{pmatrix} = \begin{pmatrix} x \\ kx \end{pmatrix}$

$R^{-1} = \frac{1}{-1} \begin{pmatrix} 12/13 & -5/13 \\ -5/13 & \frac{-12}{13} \end{pmatrix}$

$\therefore R^{-1} = \begin{pmatrix} -12/13 & 5/13 \\ 5/13 & 12/13 \end{pmatrix} = R$

Leave blank

Question 7 continued

$$\therefore \begin{pmatrix} x \\ kx \end{pmatrix} = R^{-1}\begin{pmatrix} x \\ kx \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ kx \end{pmatrix} = \begin{pmatrix} -12/13 & 5/13 \\ 5/13 & 12/13 \end{pmatrix}\begin{pmatrix} x \\ kx \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ kx \end{pmatrix} = \begin{pmatrix} -\frac{12}{13}x + \frac{5}{13}kx \\ 5/13\, x + \frac{12}{13}kx \end{pmatrix}$$

$$\therefore \quad x = x\left(\frac{5}{13}k - \frac{12}{13}\right)$$

$$\Rightarrow \quad \frac{5}{13}k - \frac{12}{13} = 1$$

$$\Rightarrow k = 5$$

$\therefore$ T maps each point ~~onto~~ on $y = kx$ onto itself for $k = 5$

Turn over

8.

$$f(z) = z^4 + 6z^3 + 76z^2 + az + b$$

where a and b are real constants.

Given that $-3 + 8i$ is a complex root of the equation $f(z) = 0$

(a) write down another complex root of this equation. **(1)**

(b) Hence, or otherwise, find the other roots of the equation $f(z) = 0$ **(6)**

(c) Show on a single Argand diagram all four roots of the equation $f(z) = 0$ **(2)**

8(a). $-3-8i$

(b) $[z-(-3+8i)][z-(-3-8i)]$

$= z^2 - z(-3-8i) - z(-3+8i) + (-3+8i)(-3-8i)$

$= z^2 + 6z + 9 - 64i^2$

$= z^2 + 6z + 73$ is a factor of $f(z)$

$\therefore f(z) \equiv (z^2+6z+73)(z^2+uz+v)$

$= z^4 + uz^3 + vz^2 + 6z^3 + 6uz^2 + 6vz$

$+ 73z^2 + 73uz + 73v$

$= z^4 + (u+6)z^3 + (v+6u+73)z^2$

$+ (6v+73u)z + 73v$

Compare coeffs: z^3

$\Rightarrow u+6 = 6 \Rightarrow u = 0$

$v = 0 \Rightarrow f(z) = z^4 + 6z^3 + (v + 73)z^2$

$+ 6vz + 73v$

Compare coeffs z^2:

$v + 73 = 76 \Rightarrow v = 3$

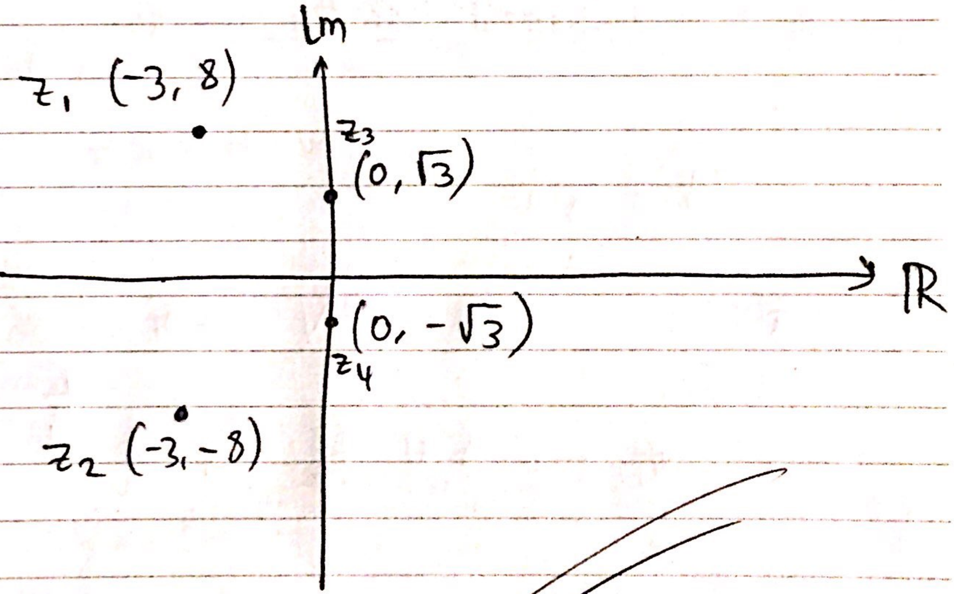
$\therefore f(z) = z^4 + 6z^3 + 76z^2 + 18z + 219$

$\Rightarrow a = 18$
$b = 219$

$z^2 + 3 = 0 \Rightarrow z = \pm i\sqrt{3}$

(c) $z^2 + 3$ is another factor

$\Rightarrow z = \pm i\sqrt{3}$



Q8

(Total 9 marks)

Leave blank

9. The quadratic equation

$$2x^2 + 4x - 3 = 0$$

has roots α and β.

Without solving the quadratic equation,

(a) find the exact value of

(i) $\alpha^2 + \beta^2$

(ii) $\alpha^3 + \beta^3$

(5)

(b) Find a quadratic equation which has roots $(\alpha^2 + \beta)$ and $(\beta^2 + \alpha)$, giving your answer in the form $ax^2 + bx + c = 0$, where a, b and c are integers.

(4)

9(a)(i) $(x-\alpha)(x-\beta) = \dfrac{2x^2+4x-3}{2} = x^2+2x-\dfrac{3}{2}$

$\therefore x^2 - \beta x - \alpha x + \alpha\beta = x^2+2x-3/2$

$\therefore x^2 - (\alpha+\beta)x + \alpha\beta = x^2+2x-3/2$

$\therefore (\alpha+\beta) = -2$

$\alpha\beta = -3/2$

$(\alpha+\beta)^2 = \alpha^2+\beta^2+2\alpha\beta$

$\therefore \alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$\therefore \alpha^2+\beta^2 = 4 - 2(-\frac{3}{2}) = 4+3 = 7$

$\therefore \alpha^2+\beta^2 = 7$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

P 4 6 6 8 3 A 0 2 4 3 2

Question 4 continued

(ii) $(\alpha+\beta)^3 = \alpha^3+\beta^3+3\alpha\beta^2+3\beta\alpha^2$

$= \alpha^3+\beta^3+3\alpha\beta(\alpha+\beta)$

$\therefore \alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$\therefore \alpha^3+\beta^3 = -8-9=-17$

(b) $[x-(\alpha^2+\beta)][x-(\beta^2+\alpha)]$

$= x^2 - x(\beta^2+\alpha) - x(\alpha^2+\beta) + (\alpha^2+\beta)(\beta^2+\alpha)$

$= x^2-(\alpha^2+\beta^2+\alpha+\beta)x + \alpha^2\beta^2 + \alpha^3+\beta^3+\alpha\beta$

$= x^2-(7-2)x + \frac{9}{4} - 17 - 3/2$

$= x^2 - 5x - \frac{65}{4} = 0$

$\therefore \quad 4x^2-20x-65=0$

$a=4$

$b=-20$

$c=-65$

Leave blank

10. (i) A sequence of positive numbers is defined by

$$u_1 = 5$$
$$u_{n+1} = 3u_n + 2, \qquad n \geqslant 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = 2 \times (3)^n - 1$$

(5)

(ii) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} \frac{4r}{3^r} = 3 - \frac{(3+2n)}{3^n}$$

(6)

10(i). When $n=1$, $u_1 = 2\times(3)^1 - 1 = 6-1 = 5$

$\therefore$ $u_1 = 5$ is correct as given.

$\therefore$ result is true for $n=1$,

When $n=k$, let us assume,

$u_k = 2\times(3)^k - 1$ is true.

Also, $n=k \Rightarrow u_{k+1} = 3u_k + 2$

$\therefore$ u_{k+1} is given by ...

$u_{k+1} = 3[2\times(3)^k - 1] + 2$

$= 6(3)^k - 1$

$= 2\cdot 3 \cdot 3^k - 1 = 2(3)^{k+1} - 1$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Leave blank

Question 10 continued

$= 2\times(3)^{k+1} - 1$ as it should for $n=k+1$ under $u_n = 2\times(3)^n - 1$

$\therefore$ Result is shown true for $n=k+1$

If the result is true for $n=k$, it is shown to be true for $n=k+1$. Since it's true for $n=1$ it must be true for $n=2,3,4,5\ldots$ and all $n\in\mathbb{Z}^+$ by induction.

(ii) When $n=1$, LHS $= \sum_{r=1}^{1} \frac{4r}{3^r} = \frac{4\times 1}{3^1} = \frac{4}{3}$

RHS $= 3 - \frac{3+2}{3} = 3 - \frac{5}{3} = \frac{4}{3}$

$\therefore$ LHS = RHS is true for $n=1$.

$\therefore$ result is true for $n=1$.

~~When~~, let's assume for $n=k$,

LHS $= \sum_{r=1}^{k} \frac{4r}{3^r} = 3 - \frac{(3+2k)}{3^k}$ = RHS

is true.

P 4 6 6 8 3 A 0 2 9 3 2

Leave blank

Question 10 continued

When $n=k+1$,

We must show

$$~~LHS~~ \sum_{r=1}^{k+1} \frac{4r}{3^r} = 3 - \frac{[3+2(k+1)]}{3^{k+1}}$$

$$= 3 - \frac{5+2k}{3^{k+1}}.$$

Now when $n=k+1$

$$LHS = \sum_{r=1}^{k+1} \frac{4r}{3^r} = \sum_{r=1}^{k} \frac{4r}{3^r} + \frac{4(k+1)}{3^{k+1}}$$

$$= 3 - \frac{(3+2k)}{3^k} + \frac{4(k+1)}{3^{k+1}}$$

$$= 3 - \frac{3(3+2k)}{3^{k+1}} + \frac{4(k+1)}{3^{k+1}}$$

$$= 3 + \frac{4k+4-9-6k}{3^{k+1}} = 3 + \frac{-2k-5}{3^{k+1}}$$

$$= 3 - \frac{5+2k}{3^{k+1}}$$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

blank

Question 10 continued

$= 3 - \frac{3+2(k+1)}{3^{k+1}}$ as it should

$\therefore$ Result is true for $n=k+1$

If the result is true for $n=k$, it is shown true for $n=k+1$, since it's true for $n=1$, it must be true for $n=2,3,4,\ldots$ and all $n \in \mathbb{Z}^+$ by induction